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Supplements to Operator-Stable and Operator-Semistable Laws on Euclidean Spaces

EBERHARD SIEBERT

Universität Tübingen, Tübingen, West Germany

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Operator-stable laws and operator-semistable laws (introduced as limit distributions by M. Sharpe and R. Jajte, respectively) are characterized by decomposability properties. Disintegration of their corresponding Lévy measures requires appropriate cross sections. Furthermore in both situations the Lévy measures constitute a Bauer simplex whose extreme boundary can be explicitly given. Finally the infinitely differentiable Lebesgue density of an operator-semistable law is shown to be even analytic in some cases. © 1986 Academic Press, Inc.

INTRODUCTION

Operator-stable probability distributions on the Euclidean space $\mathbb{R}^d$ have been introduced by M. Sharpe [12] in 1969. In 1977 R. Jajte [5] considered the more general concept of operator-semistable distributions on $\mathbb{R}^d$. These two pioneering papers have been followed by a variety of refined investigations (e.g., [4, 7, 9, 10]).

In the present paper we contribute to some questions that appeared in the papers mentioned above. The first section assembles some more or less well-known facts on the characterization of operator- (semi-) stable laws. The only new idea here may be the concept of (B, b, β)-decomposability of an infinitely divisible law on $\mathbb{R}^d$. This enabled us to give a simultaneous matrix group approach to stability and semistability.

If μ is an operator-stable law of $\mathbb{R}^d$ then there exists an automorphism A of $\mathbb{R}^d$ such that the Lévy measure η of μ satisfies $(*)$ $e^{tA}(\eta) = e^t \cdot \eta$ for all $t \in \mathbb{R}$. A detailed analysis yields that η can be disintegrated according to a

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cross section C for the orbits $\{e^{tA}x: t \in \mathbb{R}\}$, $x \in \mathbb{R}^d$ (cf. [4]). In Section 2 we show that there always exists a *closed* cross section C; in fact our construction is rather explicit. The closedness of C is important when dealing with domains of attraction, for example.

The (normalized) Lévy measure η_x that is concentrated on the orbit $\{e^{tA}x: t \in \mathbb{R}\}$ and enjoys the property (*) is obviously extremal with respect to the convex cone of all Lévy measures satisfying (*). Moreover it is plausible that every extremal Lévy measure with property (*) is of the form $c\eta_x$, $c \geq 0$. This assertion of M. Sharpe [12, pp. 61–62] has been strongly supported by the disintegration $\eta = \int_C \eta_x \, \sigma(dx)$ (of Lévy measures η satisfying (*)) due to W. N. Hudson and J. D. Mason [4, Theorem 2]. Based on this result we show in Section 3 that the set $\mathfrak{L}$ of normalized Lévy measures with property (*) is a Bauer simplex with extreme boundary $\mathfrak{L}_0 = \{\eta_x : x \in C\}$.

A similar problem arises for an operator-semistable law μ on $\mathbb{R}^d$. This time the Lévy measure ξ of μ satisfies the relation (**) $B(\xi) = \beta \cdot \xi$, where B is an automorphism of $\mathbb{R}^d$ and β a real number in $]0, 1[$. Again the (normalized) Lévy measure ξ_x concentrated on the orbit $\{B^n x: n \in \mathbb{Z}\}$ that has property (**) is easily seen to be extremal. Moreover if Z is a Borel cross section for the orbits $\{B^n x: n \in \mathbb{Z}\}$, $x \in \mathbb{R}^d$, then one has the disintegration $\xi = \int_Z \xi_x \tau(dx)$ due to A. Łuczak [9, Theorem 1.2]. Based on this result we prove in Section 4 that the set $\mathfrak{N}$ of normalized Lévy measures with property (**) is a Bauer simplex. Its extreme boundary is the set $\mathfrak{N}_0$ of all $\xi \in \mathfrak{N}$ that are concentrated on some orbit $\{B^n x: n \in \mathbb{Z}\}$ with $x \in Z$. This result strengthens an assertion of R. Jajte [5, p. 34].

Finally in Section 5 we complete a result of A. Łuczak [9, Theorem 2.2 and Remark 2.1] to the effect that the Lebesgue density of a full operator-semistable law sometimes can be extended to an analytic function in some open strip of $\mathbb{C}^d$.

Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively. Moreover let $\mathbb{Z}_+ = \{n \in \mathbb{Z}: n \geq 0\}$ and $\mathbb{R}_+^* = (x \in \mathbb{R}: x > 0\}$. If $z \in \mathbb{C}$ then $\operatorname{Re} z$ denotes its real part, $\operatorname{Im} z$ its imaginary part, and $|z|$ its absolute value.

By V we always denote the Euclidean space $\mathbb{R}^d$. Let $V^* = V \setminus \{0\}$. The scalar product in V is denoted by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $|\cdot|_2$; hence $|x|_2^2 = \langle x, x \rangle$. $U := \{x \in V: |x|_2 = 1\}$ is the unit sphere of V.

$\mathbb{M}(V)$ denotes the algebra of $(d \times d)$-matrices with real entries furnished with the operator norm $\|\cdot\|$ adapted to $|\cdot|_2$. $GL(V)$ denotes the multiplicative group of all invertible $A \in \mathbb{M}(V)$. $I \in GL(V)$ is the identity matrix. If $A \in \mathbb{M}(V)$ then $\operatorname{Spec}(A)$ denotes the set of eigenvalues and $\rho(A) = \max\{|\alpha|: \alpha \in \operatorname{Spec}(A)\}$ the spectral radius of A. We recall the formula $\rho(A)=\lim_{n\geqslant 1}\|A^n\|^{1/n}$. We use analogous notations if we consider $\mathbb{C}^d$ instead of $V=\mathbb{R}^d$.

Let X be a locally compact space with a countable basis of its topology. If Y is a subset of X then 1_Y denotes its indicator function. $\mathfrak{B}(X)$ is the σ-algebra of Borel subsets of X. Let $\mathfrak{K}(X)$ be the space of continuous real valued functions on X that have compact support. $\mathfrak{M}(X)$ denotes the space of Radon measures on X. Furnished with the vague topology, i.e., the topology of pointwise convergence on $\mathfrak{K}(X)$ it becomes a locally convex topological vector space. Since X has a countable basis of its topology the positive cone $\mathfrak{M}_+(X)$ of $\mathfrak{M}(X)$ is metrizable [2, Satz 46.4]. Let $\mathfrak{M}^1(X)=\{\mu\in\mathfrak{M}_+(X)\colon \mu(X)=1\}$ be the set of probability measures on X. Recall that $\mathfrak{M}^1(X)$ is compact if X is compact (cf. [2, Korollar 46.3]). If $x\in X$ then ε_x denotes the Dirac measure in x. Finally we recall that $\mathfrak{M}^1(V)$ is a (topological) semigroup with respect to convolution $*$.

1. Characterization by Matrix Groups

Operator-stable and operator-semistable probability distributions on the Euclidean space V have been introduced as limit laws of certain sequences of distributions and then characterized by algebraic identities [5, 12]. Turning matters around let us begin here with an algebraic definition. This yields a unified approach to stable and semistable laws.

Let $\mu\in\mathfrak{M}^1(V)$ be infinitely divisible with corresponding convolution semigroup $(\mu_t)_{t>0}$, i.e., $\mu_1=\mu$ (cf. [2, Satz 52.6]). Moreover let $B\in GL(V)$, $b\in V$, and $\beta\in\mathbb{R}_+^*$.

Definition. μ is said to be *$(B, b. \beta)$-decomposable* if the equation $\mu_\beta=B\mu * \varepsilon_b$ is valid.

Trivially every μ is $(I, 0, 1)$-decomposable. Moreover if μ is (B, b, β)-decomposable the same is true for all μ_t, $t>0$ (since μ uniquely determines its semigroup $(\mu_t)_{t>0}$).

Now let us denote by $L=L(\mu)$ the collection of all triples (B, b, β) such that μ is (B, b, β)-decomposable. Let us identify (B, b, β) with the matrix

$$\begin{pmatrix} B & b \\ 0 & \beta \end{pmatrix}\in GL(\mathbb{R}^{d+1}).$$

Then it can be easily seen that L is a closed subgroup of $GL(\mathbb{R}^{d+1})$. Hence L is a Lie group. By $f((B, b, \beta)) := \beta$ there is defined a continuous homomorphism of L into the multiplicative group $\mathbb{R}_+^*$.

Let us assume now that the measure μ is *full*, i.e., μ is not supported by a proper affine subspace of V. Then the compactness lemma of Sharpe, [12, Proposition 4] can be applied to the effect that the kernel $\{(B, b, 1) \in L: \mu = B\mu * \varepsilon_b\} =: \mathrm{Inv}(\mu)$ of f is compact and that the mapping f is closed. Hence exactly one of the following three possibilities can occur: $f(L) = \{1\}$; or $f(L) = \{\beta^n: n \in \mathbb{Z}\}$ with some $\beta \in]0, 1[$; or $f(L) = \mathbb{R}_+^*$. The case $f(L) = \{1\}$ being uninteresting let us discuss the other ones.

1. Let $f(L) = \{\beta^n: n \in \mathbb{Z}\}$ with some $\beta \in]0, 1[$. We put $k_n := [\beta^{-n}]$ for all $n \in \mathbb{N}$. Then we have $\lim k_n/k_{n+1} = \beta$ and $\lim B^n \mu^{*k_n} * \varepsilon_{b_n} = \mu$ for appropriate $b_n \in V$, $n \in \mathbb{N}$. Hence μ is operator-semistable [5, Definition 1].

2. Let $f(L) = \mathbb{R}_+^*$. Since L is a σ-compact Lie group the connected component L_0 of its identity is an open normal subgroup of L of countable index. This yields $f(L_0) = \mathbb{R}_+^*$. The image of a one-parameter subgroup of L_0 with respect to f can either be $\mathbb{R}_+^*$ or $\{1\}$. But L_0 is generated by its one-parameter subgroups. Hence there exists a one-parameter subgroup $((B(t), b(t), \beta(t)))_{t \in \mathbb{R}}$ in L_0 such that $\{\beta(t): t \in \mathbb{R}\} = \mathbb{R}_+^*$. In view of $\beta(s)\,\beta(t) = \beta(s+t)$ and $B(s)\,B(t) = B(s+t)$ for all $s, t \in \mathbb{R}$ there exist $\alpha \in \mathbb{R}^*$ and $A \in \mathbb{M}(V)$ such that $\beta(t) = e^{\alpha t}$ and $B(t) = e^{tA}$ for all $t \in \mathbb{R}$. (In fact we have $A \in GL(V)$ since μ is full.) This yields $\mu_{e^{\alpha t}} = e^{tA}\mu * \varepsilon_{b(t)}$ for all $t \in \mathbb{R}$. Putting $s_n := -\ln n/\alpha$ we get $\mu = e^{s_n A}\mu^{*n} * \varepsilon_{nb(s_n)}$ for all $n \in \mathbb{N}$. Hence μ is operator-stable ([12], p. 53).

3. Let $\mu \in \mathfrak{M}^1(V)$ be operator-semistable. Then (by definition) there exist a measure $\nu \in \mathfrak{M}^1(V)$ and sequences $(A_n)_{n \geqslant 1}$ in $GL(V)$, $(a_n)_{n \geqslant 1}$ in V, and $(k_n)_{n \geqslant 1}$ in $\mathbb{N}$ (with $k_1 = 1$) such that $(k_n)_{n \geqslant 1}$ is increasing, $\lim k_n/k_{n+1} = \beta \in]0, 1]$, and $\lim A_n \nu^{*k_n} * \varepsilon_{a_n} = \mu$. Moreover let μ be full.

(a) Let $\beta < 1$. Then there exist $B \in GL(V)$ (in fact B may be chosen as some limit point of the sequence $(A_{n+1}A_n^{-1})_{n \geqslant 1}$) and $b \in V$ such that μ is (B, b, β)-decomposable [5, Lemma 2]. Hence we have $\{\beta^n: n \in \mathbb{Z}\} \subset f(L)$.

(b) Let $\beta = 1$. Given $\beta_0 \in]0, 1[$ there exists for every $l \in \mathbb{N}$ some $n(l) \in \mathbb{N}$ such that $k_{n(l)} \leqslant \beta_0^{-l} < k_{n(l)+1}$. Then we have $\lim k_{n(l)}/k_{n(l+1)} = \beta_0$. Hence in view of (a) μ is (B_0, b_0, β_0)-decomposable with appropriate $B_0 \in GL(V)$, $b_0 \in V$. Thus $f(L) = \mathbb{R}_+^*$.

Conclusion. μ is operator-stable iff $f(L) = \mathbb{R}_+^*$; and μ is operator-semistable but not operator-stable iff $f(L) = \{\beta^n: n \in \mathbb{Z}\}$ for some $\beta \in]0, 1[$.

Remarks. 1. If μ is (B, b, β)-decomposable and full then it is easy to see that b and β are uniquely determined by B. Hence $(B, b, \beta) \to B$ yields a topological isomorphism of L onto a closed subgroup G of $GL(V)$. Sharpe has dealt with this group G in order to establish the algebraic characterization of operator-stability [12, Lemma 4]. But it appears to be more natural and effective to operate with the group L.

2. A. Łuczak has called $f(L) = \{\beta \in \mathbb{R}_+^*$: there are $B \in GL(V)$ and $b \in V$

such that $\mu_\beta = B\mu * \varepsilon_b\}$ the quasi-decomposability group of μ. In terms of $f(L)$ he has made an analysis similar to ours above (cf. [10, Theorem 3.2]). There is also a related result on locally convex topological vector spaces (cf. [3, Lemma 4]).

2. Closed Cross Sections for One-Parameter Groups of Automorphisms

When one is studying operator-stable distributions on V the following situation arises: Let $A \in GL(V)$ such that $\mathrm{Spec}(A) \subset \{z \in \mathbb{C}: \mathrm{Re}\, z > \frac{1}{2}\}$, i.e., A is the exponent of an operator-stable distribution without Gaussian component [12, Theorem 3]. There always exists a Borel cross section C for the action of $(e^{tA})_{t \in \mathbb{R}}$ on V^*, i.e., for every $x \in V^*$ the orbit $\{e^{tA}x: t \in \mathbb{R}\}$ intersects the Borel set $C \subset V^*$ at exactly one point. Consequently the mapping Φ of $C \times \mathbb{R}$ onto V^* defined by $\Phi(x, t) = e^{tA}x$ is a Borel isomorphism. Moreover Φ is continuous; and Φ is a homeomorphism if and only if C is a closed subset of V [7, Lemma I.4.3].

It has been observed by Hudson and Mason [4] that C may be chosen as a subset of the unit sphere U in V. But Kehrer has shown that their construction in general yields a non-closed cross section (cf. [7, II.4]). Certain problems, however, require a closed cross section; for example, the investigation of the domains of attraction of operator-stable distributions. Recently Jurek has proved the existence of closed cross sections even in the context of Banach spaces. In fact he shows that the unit sphere S_A with respect to the equivalent norm $\|x\|_A := \int_0^\infty |e^{-tA}x|_2 \, e^{2t}\, dt$, $x \in V$, is a closed cross section [6, Proposition 2]. But this procedure yields little information on the structure of the closed cross section. Hence we shall perform a more *explicit construction.*

1. Let us first assume that A is of the form

$$\begin{pmatrix} \alpha & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \alpha I + N$$

where α is a complex number such that $a := \mathrm{Re}\, \alpha > \frac{1}{2}$.

We fix $x \in \mathbb{C}^d \backslash \{0\}$. The function $t \to |e^{tA}x|_2^2$ is differentiable. Since

$$|e^{tA}x|_2^2 = |e^{\alpha t}\, e^{tN}x|_2^2 = e^{2at}\, |e^{tN}x|_2^2$$

we obtain

$$\begin{aligned}\frac{d}{dt}|e^{tA}x|_2^2 &= 2a\,|e^{tA}x|_2^2 + e^{2at}\,2\,\mathrm{Re}\,\langle Ne^{tN}x, e^{tN}x\rangle \\ &\geqslant 2a\,|e^{tA}x|_2^2 - e^{2at}(|Ne^{tN}x|_2^2 + |e^{tN}x|_2^2) \\ &\geqslant 2a\,|e^{tA}x|_2^2 - e^{2at}(\|N\|^2+1)\,|e^{tN}x|_2^2 \\ &= |e^{tA}x|_2^2((2a-1) - \|N\|^2).\end{aligned}$$

2. We keep the situation of part 1 and choose some $\varepsilon \in]0, (a-\frac{1}{2})^{1/2}[$. Let $D_\varepsilon := (\varepsilon^{d-i}\delta_{ij})_{1\leqslant i,j\leqslant d}$, where δ_{ij} is the Kronecker symbol. Then we have $A_\varepsilon := D_\varepsilon A D_\varepsilon^{-1} = \alpha I + \varepsilon N$ and $\|\varepsilon N\| \leqslant \varepsilon$. Applying the result of part 1 to A_ε instead of A we obtain

$$\frac{d}{dt}|e^{tA_\varepsilon}x|_2^2 \geqslant |e^{tA_\varepsilon}x|_2^2((2a-1)-\varepsilon^2) \geqslant |e^{tA_\varepsilon}x|_2^2(a-\tfrac{1}{2}).$$

3. Returning to the case of a general A there exists some $M \in GL(\mathbb{C}^d)$ such that MAM^{-1} decomposes into blocks of the form considered in part 1 (Jordan decomposition).

Putting pieces together we observe that there exist some $T \in GL(\mathbb{C}^d)$ and some $\varepsilon > 0$ such that

$$\frac{d}{dt}|e^{t(T^{-1}AT)}x|_2^2 \geqslant \varepsilon\,|e^{t(T^{-1}AT)}x|_2^2 \tag{$*$}$$

for all $x \in \mathbb{C}^d$. For example, one may choose $\varepsilon := \min\{\mathrm{Re}\,\alpha - \frac{1}{2}: \alpha \in \mathrm{Spec}(A)\}$. Since

$$|e^{t(T^{-1}AT)}x|_2 = |T^{-1}e^{tA}Tx|_2 \leqslant \|T^{-1}\|\;|e^{tA}Tx|_2$$

and since $\lim_{t\to-\infty}|e^{tA}Tx|_2 = 0$ every orbit $\{e^{t(T^{-1}AT)}x: t \in \mathbb{R}\}$ $(x \in \mathbb{C}^d\backslash\{0\})$ intersects the unit sphere $U_\mathbb{C}$ of $\mathbb{C}^d$ at some point. In view of $(*)$ this point is unique. We claim that $C := (TU_\mathbb{C}) \cap V$ is a *closed cross section* for the orbits $\{e^{tA}x: t \in \mathbb{R}\}$, $x \in V^*$.

[1. Let $x \in V^*$. Then there exists some $s \in \mathbb{R}$ such that $y := e^{s(T^{-1}AT)}(T^{-1}x) \in U_\mathbb{C}$. Hence $z := Ty \in TU_\mathbb{C}$ and $z = e^{sA}x \in V$, i.e., $e^{sA}x \in C$.

2. Let $y \in C$ and $s \in \mathbb{R}$ such that $e^{sA}y = y$. Then there exists some $z \in U_\mathbb{C}$ such that $y = Tz$. Hence $z = T^{-1}e^{sA}Tz = e^{s(T^{-1}AT)}z$ and consequently $s = 0$.]

Let $W := T^{-1}(V)$. Then W is a real linear subspace of $\mathbb{C}^d$ and $U_W := U_\mathbb{C} \cap W$ is the unit sphere of W. But we have $TU_W = TU_\mathbb{C} \cap TW = TU_\mathbb{C} \cap V = C$. Hence for an appropriate $S \in GL(V)$

the set $C=SU$ is a closed cross section. In particular C is connected if $d>1$ and C is simply connected if $d>2$.

EXAMPLE (cf. [7, II.4]). Let $d=2$ and $A=\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$. Then the ellipse $C=\{\begin{pmatrix} x \\ y-x \end{pmatrix}: x, y \in \mathbb{R}, x^2+y^2=\frac{1}{2}\}$ is a closed cross section for the orbits $\{e^{tA}x: t \in \mathbb{R}\}$ in contrast to the unit circle.

3. EXTREMAL LÉVY MEASURES OF OPERATOR-STABLE LAWS

Let $\mu \in \mathfrak{M}^1(V)$ be a full operator-stable distribution without Gaussian component and let η be the Lévy measure of μ. Then there exists some $A \in GL(V)$ such that $\operatorname{Spec}(A) \subset \{z \in \mathbb{C}: \operatorname{Re} z > \frac{1}{2}\}$ and $e^{tA}(\eta) = e^t \cdot \eta$ for all $t \in \mathbb{R}$ [12, Theorem 3; 4, Theorem 1]. Let C be a compact cross section for the orbits $\{e^{tA}x: t \in \mathbb{R}\}$, $x \in V^*$ (cf. Section 2).

As can be easily seen by $\eta'_x := \int_{\mathbb{R}} \varepsilon_{(\exp tA)x}\, e^{-t}\, dt$, $x \in V^*$, there is given a Lévy measure on V^* such that $e^{tA}(\eta'_x) = e^t \cdot \eta'_x$ for all $t \in \mathbb{R}$ (cf. [4, p. 441]). Let $\varphi(x) := |x|_2^2/(1+|x|_2^2)$ for all $x \in V^*$. Then $c_x := \int \varphi\, d\eta'_x = \int \varphi(e^{tA}x)\, e^{-t}\, dt$ is finite for all $x \in V^*$.

(i) Obviously we have $\eta'_{(\exp tA)x} = e^t \cdot \eta'_x$ and hence $c_{(\exp tA)x} = e^t\, c_x$.

(ii) $x \to c_x$ is a continuous mapping of V^* into $\mathbb{R}_+^*$.

[Choose $\delta > 0$ such that $\operatorname{Re} \alpha - \frac{1}{2} > \delta$ for all $\alpha \in \operatorname{Spec}(A)$. Then $\lim \|e^{-n(A-I/2)}\|^{1/n} = \rho(e^{-(A-I/2)}) < e^{-\delta}$. Hence there exists some $n_0 \in \mathbb{N}$ such that $\|e^{-n(A-I/2)}\| < e^{-n\delta}$ for all $n \geqslant n_0$. Put $c := \exp \|A - I/2\|$.

For every $t \in [n_0, \infty[$ there exists some $n \in \mathbb{N}$, $n \geqslant n_0$, such that $n-1 < t \leqslant n$. Consequently

$$\|e^{-t(A-I/2)}\| \leqslant e^{(n-t)\|A-I/2\|}\, \|e^{-n(A-I/2)}\| \leqslant c e^{-n\delta} \leqslant c e^{-t\delta}.$$

Moreover we have $\lim_{t \to -\infty} \|e^{tA}\| = 0$ and $\varphi(x) \leqslant \min(1, |x|_2^2)$ for all $x \in V^*$. Hence for every compact subset K of V^* and for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$, $N \geqslant n_0$, such that

$$\int_{|t|>N} \varphi(e^{tA}x)\, e^{-t}\, dt < \varepsilon \qquad \text{for all } x \in K.$$

Now let $(x_m)_{m \geqslant 1}$ be a sequence in V^* converging to $x \in V^*$. Applying the observation above to the compact set $K = \{x, x_1, x_2, \ldots\}$ we arrive at

$$|c_{x_m} - c_x| < 2\varepsilon + \int_{-N}^{N} |\varphi(e^{tA}x_m) - \varphi(e^{tA}x)|\, e^{-t}\, dt$$

for all $m \in \mathbb{N}$. Since φ is continuous we have $\overline{\lim}\, |c_{x_m} - c_x| < 2\varepsilon$ by Lebesgue's convergence theorem. Hence our assertion.]

(iii) $x \to \eta'_x$ is a (vaguely) continuous mapping of V^* into $\mathfrak{M}_+(V^*)$.

[Let $f \in \mathfrak{K}(V^*)$ and let $(x_m)_{m \geq 1}$ be a sequence in V^* converging to $x_0 \in V^*$. (Recall that $\mathfrak{M}_+(V^*)$ is (vaguely) metrizable.) Since $\{x_0, x_1, x_2, \ldots\}$ is a compact subset of V^* we have

$$\lim_{t \to -\infty} \sup\{e^{tA} x_m \|: m \in \mathbb{Z}_+\} = 0$$

and

$$\lim_{t \to \infty} \inf\{\|e^{tA} x_m\|: m \in \mathbb{Z}_+\} = \infty.$$

Consequently since f has compact support there exists some $N \in \mathbb{N}$ such that $f(e^{tA} x_m) = 0$ for all $m \in \mathbb{Z}_+$ and $t \in \mathbb{R}$ such that $|t| > N$. Hence

$$\int f\, d\eta'_{x_m} = \int_{-N}^{N} f(e^{tA} x_m)\, e^{-t}\, dt \qquad \text{for all } m \in \mathbb{Z}_+.$$

Again Lebesgue's convergence theorem yields the assertion.]

For all $x \in V^*$ let $\eta_x := c_x^{-1} \cdot \eta'_x$. In view of (i), (ii) and (iii) we have

(iv) $\eta_{(\exp tA)x} = \eta_x$ for all $x \in V^*$; and $x \to \eta_x$ is a continuous mapping of V^* into $\mathfrak{M}_+(V^*)$.

In view of (iv) for every $\sigma \in \mathfrak{M}^1(C)$ there is defined a measure $\eta_\sigma = \int_C \eta_x\, \sigma(dx)$ in $\mathfrak{M}_+(V^*)$. Obviously every η_σ belongs to the convex set $\mathfrak{L}$ of Lévy measures η on V^* such that $e^{tA}(\eta) = e^t \cdot \eta$ for all $t \in \mathbb{R}$ and $\int \varphi\, d\eta = 1$. Let $F(\sigma) := \eta_\sigma$ for all $\sigma \in \mathfrak{M}^1(C)$.

(v) F is a continuous mapping of $\mathfrak{M}^1(C)$ into $\mathfrak{L}$.

[For $f \in \mathfrak{K}(V^*)$ the function $x \to \int f\, d\eta_x$ on C is in $\mathfrak{K}(C)$ in view of (iv). Hence the assertion.]

(vi) F is an injective mapping of $\mathfrak{M}^1(C)$ onto $\mathfrak{L}$.

[Let $D \in \mathfrak{B}(C)$ and define $E = \{e^{tA} y: y \in D, 0 < t < 1\}$. Then in view of (i) we have for every $\sigma \in \mathfrak{M}^1(C)$:

$$\begin{aligned}
\int c_x\, 1_E(x)\, \eta_\sigma(dx) &= \int_C \left(\int c_x\, 1_E(x)\, \eta_y(dx) \right) \sigma(dy) \\
&= \int_C c_y^{-1} \left(\int_{\mathbb{R}} c_{(\exp tA)y}\, 1_E(e^{tA} y)\, e^{-t}\, dt \right) \sigma(dy) \\
&= \int_C \left(\int_{\mathbb{R}} 1_E(e^{tA} y)\, dt \right) \sigma(dy) = \sigma(D).
\end{aligned}$$

This proves the injectivity of F.

Now let $\eta \in \mathfrak{L}$. Then by $\nu(D) := \eta(\{e^{tA}x: x \in D, t > 0\})$, $D \in \mathfrak{B}(C)$, there is defined a finite measure ν on $\mathfrak{B}(C)$ such that $\eta = \int_C \eta'_x \nu(dx)$ [4, Theorem 2]. Denote by σ the measure on $\mathfrak{B}(C)$ with ν-density $x \to c_x$ (cf. (ii)). Then we have $\eta = \int_C \eta_x \sigma(dx)$ and $\sigma(C) = \int_C (\int \varphi \, d\eta_x) \, \sigma(dx) = \int \varphi \, d\eta = 1$, i.e., $\sigma \in \mathfrak{M}^1(C)$ and $\eta = \eta_\sigma$. This proves the surjectivity of F.]

For the theory of compact convex sets we refer to [1]. In particular let us recall that a (compact) simplex with closed extreme boundary is said to be a *Bauer simplex.*

PROPOSITION. *$\mathfrak{L}$ is a Bauer simplex and $\mathfrak{L}_0 := \{\eta_x : x \in C\}$ is its extreme boundary.*

Proof. Obviously F is an affine mapping. Since C is compact also $\mathfrak{M}^1(C)$ is compact. Hence in view of (v) and (vi) F is an affine isomorphism and a homeomorphism of $\mathfrak{M}^1(C)$ onto $\mathfrak{L}$. But $\mathfrak{M}^1(C)$ is a Bauer simplex with extreme boundary $\mathfrak{E} = \{\varepsilon_x : x \in C\}$ [1, Corollary II.4.2]. Hence $\mathfrak{L}$ is a Bauer simplex too, with extreme boundary $F(\mathfrak{E}) = \mathfrak{L}_0$. ∎

4. EXTREMAL LÉVY MEASURES OF OPERATOR-SEMISTABLE LAWS

Let $\mu \in \mathfrak{M}^1(V)$ be a full operator-semistable distribution without Gaussian component and let ξ be the Lévy measure of μ. Then there exist some $B \in GL(V)$ and $\beta \in \,]0, 1[$ such that $\mathrm{Spec}(B) \subset \{z \in \mathbb{C}: |z|^2 < \beta\}$ and $B(\xi) = \beta \cdot \xi$ [5, Theorem]. Without loss of generality let $\|B\| < 1$. Then $Z := \{x \in V: |x|_2 \leqslant 1 < |B^{-1}x|_2\}$ is a Borel cross section for the orbits $\{B^n x: n \in \mathbb{Z}\}$, $x \in V^*$. Obviously Z is a locally compact topological subspace of V^* having a countable basis of its topology. We proceed as in Section 3.

As can be easily seen by $\xi'_x := \sum_{n \in \mathbb{Z}} \beta^{-n} \varepsilon_{B^n x}$, $x \in V^*$, there is given a Lévy measure on V^* such that $B(\xi'_x) = \beta \cdot \xi'_x$ (cf. [9, p. 291]). Hence $c_x := \int \varphi \, d\xi'_x = \sum_{n \in \mathbb{Z}} \beta^{-n} \varphi(B^n x)$ is finite for all $x \in V^*$.

(i) Obviously we have $\xi'_{Bx} = \beta \cdot \xi'_x$ and hence $c_{Bx} = \beta c_x$.

(ii) $x \to c_x$ is a continuous mapping of V^* into $\mathbb{R}^*_+$.

[We have $\varphi(x) \leqslant \min(1, |x|_2^2)$ for all $x \in V^*$ and $\lim_{n \to \infty} \|B^n\|^{1/n} = \rho(B) < \sqrt{\beta} < 1$. Hence for every compact subset K of V^* and for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$\sum_{|n| > N} \beta^{-n} \varphi(B^n x) < \varepsilon \qquad \text{for all} \quad x \in K.$$

Now let $(x_m)_{m \geqslant 1}$ be a sequence in V^* converging to $x \in V^*$. Applying the observation above to the compact set $K = \{x, x_1, x_2, \ldots\}$ we arrive at

$$|c_{x_m} - c_x| < 2\varepsilon + \sum_{|n| \leqslant N} \beta^{-n} |\varphi(B^n x_m) - \varphi(B^n x)|$$

for all $m \in \mathbb{N}$. Since φ is continuous the assertion follows.]

(iii) $x \to \xi'_x$ is a continuous mapping of V^* into $\mathfrak{M}_+(V^*)$.

[Let $f \in \mathfrak{K}(V^*)$ and let $(x_m)_{m \geqslant 1}$ be a sequence in V^* converging to $x_0 \in V^*$. Then $\{x_0, x_1, x_2, \ldots\}$ is a compact subset of V^*. Moreover $\lim_{n \to \infty} \|B^n\| = 0$ in view of $\|B\| < 1$. Consequently since f has compact support there exists some $N \in \mathbb{N}$ such that $f(B^n x_m) = 0$ for all $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$ such that $|n| > N$. Hence

$$\int f \, d\xi'_{x_m} = \sum_{|n| \leqslant N} \beta^{-n} f(B^n x_m) \qquad \text{for all } m \in \mathbb{Z}_+.$$

Since f is continuous the assertion follows.]

For all $x \in V^*$ let $\xi_x := c_x^{-1} \cdot \xi'_x$. In view of (i), (ii) and (iii) we have

(iv) $\xi_{Bx} = \xi_x$ for all $x \in V^*$; and $x \to \xi_x$ is a continuous mapping of V^* into $\mathfrak{M}_+(V^*)$.

In view of (iv) for every $\tau \in \mathfrak{M}^1(Z)$ there is defined a measure $\xi_\tau = \int_Z \xi_x \, \tau(dx)$ in $\mathfrak{M}_+(V^*)$. Obviously every ξ_τ belongs to the convex set $\mathfrak{N}$ of Lévy measures ξ on V^* such that $B(\xi) = \beta \cdot \xi$ and $\int \varphi \, d\xi = 1$. Let $G(\tau) := \xi_\tau$ for all $\tau \in \mathfrak{M}^1(Z)$.

(v) G is a continuous mapping of $\mathfrak{M}^1(Z)$ into $\mathfrak{N}$.

[For every $f \in \mathfrak{K}(V^*)$ the function $x \to \int f \, d\xi_x$ on Z is bounded (since $|f| \leqslant c \cdot \varphi$ for some $c > 0$) and continuous (in view of (iv)). Hence the assertion.]

(vi) G is an injective mapping of $\mathfrak{M}^1(Z)$ onto $\mathfrak{N}$.

[If $x \in Z$ then the restriction of ξ_x to Z is just $c_x^{-1} \cdot \varepsilon_x$ (cf. [9, Lemma 1.2]). Hence for every $f \in \mathfrak{K}(Z)$ and $\tau \in \mathfrak{M}^1(Z)$ we have

$$\begin{aligned} \int_Z c_y f(y) \, \xi_\tau(dy) &= \int_Z \left(\int_Z c_y \, f(y) \, \xi_x(dy) \right) \tau(dx) \\ &= \int_Z f(x) \, \tau(dx). \end{aligned}$$

This proves the injectivity of G.

Now let $\xi \in \mathfrak{N}$ and denote by ν the restriction of ξ to Z. Then we have $\xi = \int_Z \xi'_x \, \nu(dx)$ [9, Theorem 1.2]. Denote by τ the measure on $\mathfrak{B}(Z)$ with

ν-density $x \to c_x$ (cf. (ii)). Then we have $\xi = \int_Z \xi_x \tau(dx)$ and $\tau(Z) = \int_Z (\int \varphi \, d\xi_x) \tau(dx) = \int \varphi \, d\xi = 1$, i.e., $\tau \in \mathfrak{M}^1(Z)$ and $\xi = \xi_\tau$. This proves the surjectivity of G.]

Proposition. *$\mathfrak{N}$ is a Bauer simplex and $\mathfrak{N}_0 := \{\xi_x : x \in Z\}$ is its extreme boundary.*

Proof. 1. $\mathfrak{N}_0$ is (vaguely) compact.

[The closure $\bar{Z} = \{x \in V^*: |x|_2 \leqslant 1 \leqslant |B^{-1}x|_2\}$ of Z is compact. If $x \in \bar{Z} \setminus Z$ we must have $|B^{-1}x|_2 = 1$ and hence $y := B^{-1}x \in Z$. In view of (iv) this yields $\xi_x = \xi_{By} = \xi_y \in \mathfrak{N}_0$. Hence $x \to \xi_x$ maps $\bar{Z}$ onto $\mathfrak{N}_0$. Since this mapping is continuous in view of (iv) the assertion follows.]

2. Let $h(x) := \xi_x$ for all $x \in Z$. In view of (iv) and (vi) h is a continuous bijection of Z onto $\mathfrak{N}_0$. Since Z is σ-compact h is a Borel isomorphism. Hence h extends to an affine isomorphism H of $\mathfrak{M}^1(Z)$ onto $\mathfrak{M}^1(\mathfrak{N}_0)$. Taking into account (vi) the mapping $I := G \circ H^{-1}$ is an affine isomorphism of $\mathfrak{M}^1(\mathfrak{N}_0)$ onto $\mathfrak{N}$.

3. Let $f \in \mathfrak{K}(V^*)$ and $\kappa \in \mathfrak{M}^1(\mathfrak{N}_0)$. Then an easy calculation yields

$$\int_{V^*} f \, dI(\kappa) = \int_{\mathfrak{N}_0} \left(\int_{V^*} f \, d\xi \right) \kappa(d\xi).$$

But $\mathfrak{N}_0 \ni \xi \to \int_{V^*} f \, d\xi$ is continuous and hence bounded (since $\mathfrak{N}_0$ is compact in view of part 1). Hence I is a continuous bijection of $\mathfrak{M}^1(\mathfrak{N}_0)$ onto $\mathfrak{N}$ and thus a homeomorphism (since $\mathfrak{M}^1(\mathfrak{N}_0)$ is compact).

Now $\mathfrak{M}^1(\mathfrak{N}_0)$ is a Bauer simplex with extreme boundary $\mathfrak{F} = \{\varepsilon_{\xi_x} : x \in Z\}$ [1, Corollary II.4.2]. Hence $\mathfrak{N}$ is a Bauer simplex too with extreme boundary $I(\mathfrak{F}) = \mathfrak{N}_0$. ∎

5. Analyticity of Densities of Operator-Semistable Laws

Let $\mu \in \mathfrak{M}^1(V)$ be a full operator-semistable distribution. In view of the decomposition result of Jajte [5, Theorem] and in view of the well-known analytical properties of Gaussian laws we may assume without loss of generality that μ has no Gaussian component (see also [9, p. 294]). We recall that then there exist $\beta \in]0, 1[$, $B \in GL(V)$ such that $\mathrm{Spec}(B) \subset \{z \in \mathbb{C}: |z|^2 < \beta\}$, and $b \in V$ such that μ is (B, b, β)-decomposable (cf. [5]). Without loss of generality let $\|B\| < 1$ (cf. [9, Remark 2.1]).

It has been proved by Łuczak [9, Theorem 2.2 and Remark 2.1] that μ admits an infinitely differentiable density f with respect to the Lebesgue measure λ on V. We will show now that Łuczak's method yields sometimes even a sharper result.

PROPOSITION. *Let* $|\alpha| \geqslant \beta$ *for all* $\alpha \in \mathrm{Spec}(B)$.

(i) *If* $|\alpha| > \beta$ *for all* $\alpha \in \mathrm{Spec}(B)$ *then the density* f *of* μ *can be extended to an entire function (on* $\mathbb{C}^d$*).*

(ii) *If every* $\alpha \in \mathrm{Spec}(B)$ *with* $|\alpha| = \beta$ *is a simple root of the minimal polynomial then* f *can be extended to an analytic function in some strip* $S_\delta := \{(z_1, ..., z_d) \in \mathbb{C}^d : |\mathrm{Im}\, z_j| < \delta \text{ for } j = 1, ..., d\}$ $(\delta > 0)$.

Proof. (i) There exists some $c \in]\frac{1}{2}, 1[$ such that $\beta^c < |\alpha|$ for all $\alpha \in \mathrm{Spec}(B)$. Taking into account the proof of Theorem 2.1 in [9] there exist $a, b > 0$ such that $|\hat{\mu}(x)| \leqslant 1$ if $|x|_2 \leqslant a$ and $|\hat{\mu}(x)| \leqslant \exp\{-b\,|x|_2^{1/c}\}$ if $|x|_2 > a$ (where $\hat{\mu}$ denotes the Fourier transform of μ). Let $\varepsilon := 1/c - 1 > 0$. Fix $\delta > 1$ and let $x \in V$ with $ix \in S_{\delta/d}$. Then we have for all $y \in V$ such that $|y|_2 > a$:

$$\begin{aligned} e^{\langle x, y\rangle}\, |\hat{\mu}(y)| &\leqslant \exp\{\delta\, |y|_2 - b\, |y|_2^{1+\varepsilon}\} \\ &= \exp\{-|y|_2 (b\, |y|_2^{\varepsilon} - \delta)\}. \end{aligned}$$

Hence

$$e^{\langle x, y\rangle}\, |\hat{\mu}(y)| \leqslant \exp\{-|y|_2\}$$

for all $y \in V$ such that $|y|_2 > \max\{a, ((\delta - 1)/b)^{1/\varepsilon}\}$. Thus the function $y \to e^{\langle x, y\rangle}\, \hat{\mu}(y)$ on V is λ-integrable. Application of [2, Satz 49.5] to the measure with λ-density $\hat{\mu}$ yields:

$$w \to \int e^{-i\langle w, y\rangle}\, \hat{\mu}(y)\, \lambda(dy)$$

is an analytic function F on $S_{\delta/d}$. But in view of the inversion formula of Fourier analysis we have $F(x) = f(x)$ for all $x \in V$. Since $\delta > 1$ was arbitrary the first assertion is proved.

(ii) Let there exist some $\alpha \in \mathrm{Spec}(B)$ such that $|\alpha| = \beta$. Let B^t denote the transpose of B and let $A := \beta (B^t)^{-1}$. By assumption $\rho(A) = 1$ and every $\alpha \in \mathrm{Spec}(A)$ with $|\alpha| = 1$ is a simple root of the minimal polynomial. Hence there exists some $c > 0$ such that $\|A^n\| \leqslant c$ for all $n \in \mathbb{N}$ [11, p. 43, Exercise 6].

Let $Z := \{x \in V : |x|_2 \leqslant 1 < |B^{-1}x|_2\}$ and $u := -\ln |\hat{\mu}|$. Then there exists some $\varepsilon > 0$ such that $u(x) \geqslant \varepsilon$ for all $x \in Z$ (cf. [9, p. 295]). Moreover we have $u((B^t)^n x) = \beta^n u(x)$ for all $x \in Z$ and $n \in \mathbb{Z}$. Consequently we obtain for all $x \in Z$ and $n \in \mathbb{N}$:

$$\frac{u((B^t)^{-n}x)}{|(B^t)^{-n}x|_2} = \frac{\beta^{-n}\, u(x)}{|(B^t)^{-n}x|_2} = \frac{u(x)}{|A^n x|_2} \geqslant \frac{u(x)}{\|A^n\|\, |x|_2} \geqslant \frac{\varepsilon}{c} =: b.$$

Hence $u(y) \geqslant b\,|y|_2$ for all $y \in \bigcup_{n \geqslant 1} (B')^{-n}Z$. On the other hand we have $|y|_2 \leqslant 1$ for all $y \in \bigcup_{n \leqslant 0} (B')^{-n}Z$ (since $\|B\| < 1$). Since $\bigcup_{n \in \mathbb{Z}} (B')^n Z = V^*$ [9, Corollary 1.1] we obtain $|\hat{\mu}(x)| \leqslant 1$ if $|x|_2 \leqslant 1$ and $|\hat{\mu}(x)| \leqslant \exp\{-b\,|x|_2\}$ if $|x|_2 > 1$.

Let $\delta \in]0, b[$ and $x \in V$ such that $ix \in S_{\delta/d}$. Then we have for all $y \in V$ such that $|y|_2 > 1$:

$$e^{\langle x, y \rangle}\,|\hat{\mu}(y)| \leqslant \exp\{-(b-\delta)\,|y|_2\}.$$

Now we can proceed as in the proof of (i). Hence f admits an analytic extension to the strip $S_{b/d}$. ▌

Remarks. 1. The case (i) of the proposition arises if and only if μ admits an absolute moment of order 1, i.e., iff $\int |x|_2\,\mu(dx)$ is finite [9, Theorem 3.1].

2. On the real line this proposition is due to V. M. Kruglov [8, Theorem 3]. It should be observed that in this case the matrix B can be identified with a real number $B \in [\beta, \sqrt{\beta}[$. Then $\beta = B^\alpha$ with some $\alpha \in [1, 2[$. Hence if $\alpha > 1$ then f can be extended to an entire function; if $\alpha = 1$ then f can be extended to an analytic function in some strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$.

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